ISRAEL JOURNAL OF MATHEMATICS, Vol. 61, No. 2, 1988

INVARIANT ADDITIVE SUBGROUPS OF A CERTAIN KIND

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ABSTRACT

Let R be a prime ring with a nonzero nil right ideal, and let M be the union of all nil right ideals of R. If W is an additive subgroup of R which is invariant under conjugation by all special automorphisms $1+x$ for $x \in M$, then either W is central or W contains a noncommutative Lie ideal of R. Assuming that W is invariant under only those $1+x$ for $x \in M$ and $x^2=0$, the same conclusion holds if the extended centroid of R is not GF(2).

This paper extends a result of I. N. Herstein [6] on the structure of invariant subrings of a certain class of prime rings, to the case of invariant additive subgroups. A subset S of a ring R is invariant with respect to a set M of nilpotent elements of R if for each $t \in S$ and $x \in M$,

$$(1-x)t(1-x)^{-1} = t - xt + t\bar{x} - xt\bar{x} \in M, \qquad \text{where } \bar{x} = x + x^2 + \cdots.$$

Until the appearance of [6], and except for division rings, the most general results on the structure of invariant subrings follow from work of S. A. Amitsur [1]. He proves that for R a simple ring with nontrivial idempotent and centroid $Z \neq \mathrm{GF}(2)$, any subalgebra of R invariant with respect to $T = \{x \in R \mid x^2 = 0\}$ is 0, Z, or R; and any Z-subspace invariant under T must be central or contain $[R, R]$. The arguments in [1] are fairly easy and readily extend to prime rings, but for this extension it is crucial that one works with subspaces and have an idempotent in R. In [6], Herstein obtains a dichotomy theorem for subrings, rather than for subalgebras, and does not require an idempotent, although a rather restricted class of rings is considered. For later reference we state his theorem now [6; Theorem, p. 206].

Received August 4, 1987

THEOREM. *Let R be prime ring containing a nonzero nil right ideal, and let M be the union of all nil right ideals of R. If W is a subring of R invariant with respect to M, then either W is central or W contains a nonzero ideal of R.*

By using methods generally similar to those required for this theorem, Herstein improved Amitsur's result to subrings of prime rings and essentially eliminated the condition that $Z \neq \mathrm{GF}(2)$ [7]. Subsequently, this work of Herstein was used by C. L. Chuang [3] to obtain the version of Amitsur's result for invariant additive subgroups. Although these later results still require the presence of an idempotent in an essential way, their general approach, together with an adaptation of arguments used in [6], leads to our extension of Herstein's Theorem stated above. Our main result shows that an additive subgroup invariant with respect to M either is central or contains a Lie ideal of R. Furthermore, if one assumes invariance only with respect to the elements of square zero in M, then the same conclusion holds if the extended centroid of R is not GF(2).

Throughout the paper, R will denote a prime ring containing a nonzero nil right ideal. Let M be the set-theoretic union of all nil right ideals in R and observe that if $x \in M$ and $r, s \in R$, then $rx, xs, rxs \in M$. If follows that V, the additive subgroup of R generated by M, is an ideal of R. For any $S \subset R$, set $T(S) = \{x \in S \mid x^2 = 0\}$. Finally, Z will denote the centroid of R and C will denote the extended centroid of R (see [9] or [5]).

Our first results are technical but useful lemmas. They correspond to various observations and claims proved in [6]. The complications here are not only because we consider subgroups rather than subrings, but also because we want to prove our results for invariance under $T(M)$ rather than under all of M.

LEMMA 1. *If $t \in T(M)$ and $tT(M)t = 0$, then $t = 0$.*

PROOF. It is clear that $T(M)$ is invariant with respect to M. Now for any $y \in M$ and $r \in R$, since $trt \in T(M)$, we have $(1-y)trt(1-y)^{-1} \in T(M)$. Thus, $t(1-y)trt(1-y)^{-1}t = 0$ and it follows from the primeness of R that either $tyt = 0$ or $t(1-y)^{-1}t = t(y + y^2 + \cdots + y^n)t = 0$, where $y^{n+1} = 0$. An easy argument by induction on n shows that $tyt = 0$, from which $tVt = 0$ results. Therefore $t = 0$, using the primeness of R again.

LEMMA 2. *Let W be a nonzero additive subgroup of R invariant with respect to $T(M)$. If $x \in R$ and either $xW = 0$ or $Wx = 0$ then $x = 0$.*

PROOF. We consider the case $xW = 0$. For any $w \in W$ and $t \in T(M)$, $x(1+t)w(1+t)^{-1} = 0$, and so $xtw = 0$, forcing $xT(M)w = 0$. Choose $y \in M$ and note that $wyx \in M$, so $xw = 0$ yields $wyx \in T(M)$. But now $wyxT(M)wyx = 0$ and we must conclude that $wyx = 0$ by Lemma 1. Consequently, $wMx = 0$, and so $wVx = 0$, resulting in $w = 0$ or $x = 0$. Since $W \neq 0$, $x = 0$ follows. A similar argument shows that $Wx = 0$ implies $x = 0$.

Our next result requires the use of Herstein's Theorem.

LEMMA 3. *If $S = \{r \in R \mid trt = 0$ for all $t \in T(M)\}$ then $S \subset Z$.*

PROOF. It is clear that S is an additive subgroup of R. Also S is invariant with respect to M since $T(M)$ is. We claim that S is a subring. Let $x \in S$ and $t \in T(M)$. Then $trt \in T(M)$ for any $r \in R$ and $trtx \in T(M)$, because $txt = 0$. Consequently $(trtx)s(trtx) = 0$ for $s \in S$, and it follows that $txstR$ is a nil right ideal of R of index 3. The primeness of R and Levitzki's Theorem [4; Lemma 1.1, p. 1] yield $txst = 0$, and so, $S^2 \subset S$. Thus S is a subring and we may apply Herstein's Theorem to S. If $S \supset I \neq 0$ an ideal of R, then $tIt = 0$ for $t \in T(M)$ would force $T(M) = 0$. Since $T(M) \neq 0$ we must have $S \subset Z$, proving the lemma.

For various computations which will follow it is convenient to set

$$F(t, w) = (1+t)w(1+t)^{-1} - w \qquad \text{for } w \in R \text{ and } t \in M.$$

Note that $F(t, w) = [t, w](1+t)^{-1}$, where $[t, w] = tw - wt$, and when $t \in T(M)$, $F(t, w) = [t, w] - twt$. Finally if W is an additive subgroup of R invariant with respect to $\{-t\}$, then $F(t, w) \in W$.

The next lemma shows when an invariant subgroup is central, and requires the notion of the Martindale quotient ring (see [5] or [9]).

LEMMA 4. *Let W be an additive subgroup of R invariant with respect to $T(M)$. If $W \cap T(M) = 0$ then $W \subset Z$.*

PROOF. Let $t \in T(M)$, $r \in R$, and $w \in W$, so also $trt \in T(M)$. Consequently, $F(t, w) = tw - wt - twt \in W$, and so, $F(trt, F(t, w)) = -(trtwt + twtrt) \in W \cap T(M)$. Hence $trtwt + twtrt = 0$ for each $r \in R$, and it follows from a result of Martindale [5; Lemma 1.3.2, p. 22] that $twt = ct$ for some $c \in C$, the extended centroid of R. Now $twtw = ctw$, so $(tw)^n = c^{n-1}tw$ for any $n > 1$, but $tw \in M$, so is nilpotent. Therefore $c = 0$ or $tw = 0$, so in either case $twt = 0$ for each $w \in W$ and $t \in T(M)$. Lemma 3 now yields $W \subset Z$.

Our last preliminary lemma shows that an invariant subgroup is "semiprime".

LEMMA 5. *Let W be an additive subgroup of R invariant with respect to $T(M)$. If $x \in T(W)$ and $xWx = 0$ then $x = 0$.*

PROOF. As we have seen, $trt \in T(M)$ for $t \in T$ and $r \in R$. Thus $0 = xF(trt, x)x = -xtrtxtrtx$, so $txtR$ is a nil right ideal of R of index 3, and Levitzki's Theorem [4; Lemma 1.1, p. 1] yields $txt = 0$. Consequently, $x = 0$ by Lemma 3.

We now have all the pieces needed to prove our first main theorem, which is a direct extension of Herstein's Theorem. The argument follows his but is modified to deal with additive subgroups. The conclusion of our theorem mentions Lie ideals, which are additive subgroups L of R so that $[x, r] = xr - rx \in L$ for all $x \in L$ and $r \in R$. A noncentral Lie ideal L of R can be commutative, that is $[L, L] = 0$, only if R satisfies a polynomial identity [8; Theorem 4, p. 118], which in turn would force any nil right ideal of R to be nilpotent [4; Lemma 5.4, p. 91]. Since R is a prime ring, this situation is not possible if R contains a nonzero nil right ideal. Consequently, with our assumptions on R, any noncentral Lie ideal of R is noncommutative, and so must contain a Lie ideal of the form $[I, R]$ for I a nonzero ideal of R [8; Theorem 13, p. 123].

THEOREM 1. *Let W be an additive subgroup of R invariant with respect to M. Then either $W \subset Z$ or $W \supset [I, R]$ for I a nonzero ideal of R.*

PROOF. If $W \cap T(M) = 0$, then $W \subset Z$ by Lemma 4. Hence, we may assume there is a nonzero $w \in W \cap T(M)$. For $x \in M$ we have wx and xw in M, so $F(xw, w) = -wxw \in W$, and it follows first that $wMw \subset W$, and then that $wVw \subset W$. Choose $a \in W$ and consider $F(wx, a) = [wx, a](1 + wx)^{-1} \in W \cap V$. Since $w \in T(M)$,

$$b = F(w, F(wx, a)) = [w, [wx, a](1 + wx)^{-1}](1 - w) \in W.$$

Expanding gives

$$b = w[wx, a](1 + wx)^{-1} - [wx, a](1 + wx)^{-1}w - wF(wx, a)w,$$

and because $F(wx, a) \in V$ and $wVw \subset W$, we obtain

$$w[wx, a](1 + wx)^{-1} - [wx, a](1 + wx)^{-1}w \in W.$$

Using $w^2 = 0$ results in

$$-wawx(1+wx)^{-1} - wxa(1+wx)^{-1}w + awx(1+wx)^{-1}w \in W.$$

Now $x \in M \subset V$, so $w(xa(1+wx)^{-1})w \in wVw \subset W$ and we may finally conclude that $[awx(1+wx)^{-1}, w] \in W$.

Next, observe that for any $y \in M$, we may write $y = x(1+wx)^{-1}$ for $x = (1-yw)^{-1}y \in M$. Using this x in $[awx(1+wx)^{-1}, w] \in W$ results in $[awM, w] \subset W$, and then since W is an additive subgroup, $[awV, w] \subset W$. The quasi-inverse of xw is

$$(xw)' = -xw + (xw)^2 - \cdots = xt \in M, \qquad \text{for some } t \in R.$$

Thus, because W is invariant with respect to M,

$$\bar{a} = (1+xw)^{-1}a(1+xw) - a = (1+(xw)')a(1+(xw)')^{-1} - a \in W$$

and

$$\bar{a} = (1+xw)^{-1}[a, xw].$$

Replacing a above with $\bar{a}$ gives

$$[\bar{a}wV, w] = [(1+xw)^{-1}[a, xw]wV, w] = [(1+xw)^{-1}xwawV, w] \subset W.$$

As above, any $y \in M$ can be written as $y = (1+xw)^{-1}x$ for $x = y(1-wy)^{-1} \in M$, and we obtain first that $[MwawV, w] \subset W$, and then that $[VwawV, w] \subset W$. If $VwWwV = 0$ then $wWw = 0$ follows from the primeness of R, and then Lemma 5 would force $w = 0$. This contradiction shows that for some $a \in W$, $I = VwawV$ is a nonzero ideal of R.

Now set $B = \{r \in R \mid [I, r] \subset W\}$ and observe that $B \not\subset Z$ since $w \in B \cap T(M)$. Clearly, B is an additive subgroup of R and is invariant with respect to M since both W and I are. Furthermore, the identity $[a, bc] = [ab, c] + [ca, b]$ shows that B is a subring of R. Therefore, we may apply Herstein's Theorem to B and conclude that $J \subset B$ for a nonzero ideal J of R. Consequently, W contains the Lie ideal $L = [I, J]$ of R and the proof is complete by our comments above, since L is not central [8; Lemma 7, p. 120].

We state a simple consequence of Theorem 1 which will be important for our last result on invariance under $T(M)$.

THEOREM 2. *If I is a nonzero ideal of R, then $(I \cap T(M), +)$ the additive subgroup of R generated by $I \cap T(M)$, contains a noncommutative Lie ideal of R. In particular, $(T(M), +)$ contains a noncommutative Lie ideal of R.*

PROOF. Clearly $(I \cap T(M), +)$ is invariant with respect to M, and since for $t \in T(M) - \{0\}$, $tIt \subset I \cap T(M)$, $I \cap T(M) \not\subset Z$. The conclusion now follows from Theorem 1.

Our interest in Theorem 2 is that it enables us to exploit the basic technique of Baxter [2], used by Amitsur [1], to consider subgroups invariant under $T(M)$. The observation is that for $t \in T(M)$, $w \in W$, and $c \in Z$, $F(ct, w) - c^2F(t, w) = [(c - c^2)t, w]$. Thus, if $c^2 \neq c$ and $c^2F(t, w) \in W$, then one has $[(c - c^2)(T(M), +), w] \subset W$. Since Theorem 2 implies that $(T(M), +)$ contains a Lie ideal, we are almost in the situation of $[L, W] \subset W$ for a Lie ideal L, and the Lie structure theory in [8] would apply. Our next result shows how to overcome the technical difficulties which arise in the approach just mentioned, although we cannot eliminate the constraint that $C \neq \mathrm{GF}(2)$.

THEOREM 3. *Let W be an additive subgroup of R invariant with respect to $T(M)$. If $C \neq \mathrm{GF}(2)$ then either $W \subset Z$ or $W \supset [I, R]$ for I a nonzero ideal of R. In addition, if W is a subring then either $W \subset Z$ or W contains a nonzero ideal of R.*

PROOF. The statement for W a subring follows from the case when W is an additive subgroup and the fact that the subring generated by a noncommutative Lie ideal of R contains a nonzero ideal of R [4; proof of Lemma 1.3, p. 4]. Hence, we need only assume that W is an additive subgroup. Choose $c \in C$ so that $c^2 \neq c$ and let I be a nonzero ideal of R satisfying $cI + c^2I \subset R$ ([9; p. 577] or [5, pp. 21–22]). For any $t \in T(M)$, $x \in I$, and $w \in W$, both $txt \in T(M)$ and $c^2txt \in T(M)$, so $F(-t, F(txt, w)) = txtwt + twtxt \in W$, and also, $c^2(txtwt + twtxt) = F(-t, F(tc^2xt, w)) \in W$. Thus $txtwt + twtxt \in U = \{w \in W \mid c^2w \in W\}$, which is clearly an additive subgroup of R invariant with respect to $T(M)$, since W is. Now if $U = 0$ then the argument of Lemma 4 shows that $W \subset Z$, completing the proof. Therefore, we may assume that $U \neq 0$.

For any $t \in T(M) \cap I$, $ct \in R$ and ct generates a nil right ideal, so $ct \in T(M)$. Hence if $u \in U$, since $F(t, u) \in U$, we have $F(ct, u) - c^2F(t, u) = (c - c^2)[t, u] \in W$. It follows that $[(T(M) \cap (c - c^2)I +), U] \subset W$, so by Theorem 2 and the comments preceding it, $[[J, R], U] \subset W$ for J a nonzero ideal of R. But now,

$$c^2[[J, I], U] = [[J, c^2I], U] \subset [[J, R], U] \subset W,$$

which shows that $[[J, I], U] \subset U$. Since $[J, I]$ is a noncentral Lie ideal of R [8; Lemma 7, p. 120], we may apply [8; Theorem 13, p. 123] to conclude that $U \subset Z$ or $L \subset U \subset W$ for L a noncommutative Lie ideal of R. From above there is a nonzero $twtxt + txtwt \in U \cap T(M)$, so $U \subset Z$ is impossible and the theorem is proved.

An easy and straightforward adaptation of the proof of Theorem 3 yields the following slight generalization.

THEOREM 4. *Let I be a nonzero ideal of R and W an additive subgroup of R invariant with respect to $I \cap T(M)$. If $C \neq$ GF(2) then either $W \subset Z$ or W contains a noncommutative Lie ideal of R. If W is a subring then $W \subset Z$ or W contains a nonzero ideal of R.*

REFERENCES

1. S. A. Amitsur, *Invariant submodules of simple rings*, Proc. Am. Math. Soc. 7 (1956), 987–989.
2. W. E. Baxter, *Lie simplicity of a special class of associative rings*, Proc. Am. Math. Soc. 7 (1956), 855–863.
3. C. L. Chuang, *On invariant additive subgroups*, Isr. J. Math. **57** (1987), 116–128.
4. I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago, 1969.
5. I. N. Herstein, *Rings with Involution*, University of Chicago Press, Chicago, 1976.
6. I. N. Herstein, *Invariant subrings of a certain kind*, Isr. J. Math. **26** (1977), 205–208.
7. I. N. Herstein, *A theorem on invariant subrings*, J. Algebra **83** (1983), 26–32.
8. C. Lanski and S. Montgomery, *Lie structure of prime rings of characteristic* 2, Pacific J. Math. **42** (1972), 117–136.
9. W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.